MATH2050B Mathematical Analysis I

Midterm Test 2 suggested Solution*

Question 1. Let $I(\neq \emptyset)$ be an interval and $f : I \to \mathbb{R}$. State the definitions/notations:

(i) f is continuous (cts) at $x_0 \in I$.

(ii) f is uniformly continuous on I.

State the negation for each of (i), (ii).

Solution:

(i) We say that f is continuous at x_0 if, given any number $\varepsilon > 0$, there exists $\delta(\varepsilon, x_0) > 0$ such that if x is any point of I satisfying $|x - x_0| < \delta(\varepsilon, x_0)$, then $|f(x) - f(x_0)| < \varepsilon$.

Negation: There exists $\varepsilon_0 > 0$ such that for any $\delta > 0$, there exists $x' \in I$ satisfying $|x' - x_0| < \delta$, such that $|f(x') - f(x_0)| \geq \varepsilon_0$.

(ii) We say that f is uniformly continuous on I if for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $x, y \in I$ are any numbers satisfying $|x - y| < \delta(\varepsilon)$, then $|f(x) - f(y)| < \varepsilon$.

Negation: There exists an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are points x_δ, y_δ in I such that $|x_\delta - y_\delta| < \delta$ and $|f(x_\delta) - f(y_\delta)| \geq \varepsilon_0$.

The negation can be alternatively be stated: There exists an $\varepsilon_0 > 0$ and two sequences (x_n) and (y_n) in I such that $\lim (x_n - y_n) = 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

Question 2. In the terminology of $\varepsilon - \delta$, show that

$$\lim_{x \to 3} \frac{x^2 + 7}{x - 2} = 16.$$

Solution: Let $\varepsilon > 0$, take $\delta(\varepsilon) = \min\{\frac{1}{2}, \frac{\varepsilon}{35}\}$.

Suppose $|x - 3| < \delta(\varepsilon)$, then

$$-\frac{1}{2} < x - 3 < \frac{1}{2} \quad \text{i.e.} \quad \frac{5}{2} < x < 4,$$

which implies that $x - 2 > \frac{1}{2}$ and $|x - 13| < \frac{21}{2}$.

*please kindly send an email to cyma@math.cuhk.edu.hk if you have any question.

It follows that

$$\begin{aligned}\left|\frac{x^2+7}{x-2}-16\right| &= \left|\frac{x^2+7-16x+32}{x-2}\right| \\ &= \left|\frac{x^2-16x+39}{x-2}\right| \\ &= \frac{|x-3|\cdot|x-13|}{|x-2|} \\ &< \frac{|x-3|\cdot|x-13|}{\frac{1}{2}} \\ &< \frac{\delta(\varepsilon)\cdot 17}{\frac{1}{2}} \\ &= 34\delta(\varepsilon) \\ &< \varepsilon.\end{aligned}$$

Therefore $\lim_{x\to 3}\frac{x^2+7}{x-2}=16$.

Question 3. Let $I=[a,b]$ and suppose that f is cts on $[a,b]$. Show that

(i) f is uniformly cts on $[a,b]$.

(ii) If $f(a)>0>f(b)$ then $\exists\ x_0\in(a,b)$ such that $f(x_0)=0$.

Solution:

(i) *Proof.* Suppose on the contrary that f is not uniformly continuous on I. Then, by Q1(ii), there exist $\varepsilon_0>0$ and two sequences (x_n) and (y_n) in I such that $|x_n-y_n|<1/n$ and $|f(x_n)-f(y_n)|\geq\varepsilon_0$ for all $n\in\mathbb{N}$. Since I is bounded, the sequence (x_n) is bounded; by the Bolzano-Weierstrass Theorem, there is a subsequence (x_{n_k}) of (x_n) that converges to an element z. Since $a\leq x_{n_k}\leq b$ for all $n\in\mathbb{N}$, we obtain $a\leq\lim x_{n_k}\leq b$. It follows that the limit z belongs to I. It is clear that the corresponding subsequence (y_{n_k}) also converges to z, since

$$|y_{n_k}-z|\leq|y_{n_k}-x_{n_k}|+|x_{n_k}-z|.$$

Now if f is continuous at the point z, then both of the sequences $(f(x_{n_k}))$ and $(f(y_{n_k}))$ must converge to $f(z)$. But this is not possible since

$$|f(x_n)-f(y_n)|\geq\varepsilon_0$$

for all $n\in\mathbb{N}$. Thus the hypothesis that f is not uniformly continuous on the closed bounded interval I implies that f is not continuous at some point $z\in I$. Consequently, if f is continuous at every point of I, then f is uniformly continuous on I.

(ii) **Method 1:**

Proof. We will generate a sequence of intervals by successive bisections. Let $I_1:=[a_1,b_1]$, where $a_1:=a, b_1:=b$, and let p_1 be the midpoint $p_1:=\frac{1}{2}(a_1+b_1)$. If $f(p_1)=0$, we take $x_0:=p_1$

and we are done. If $f(p_1) \neq 0$, then either $f(p_1) > 0$ or $f(p_1) < 0$. If $f(p_1) > 0$, then we set $a_2 := p_1, b_2 := b_1$, while if $f(p_1) < 0$ then we set $a_2 := a_1, b_2 := p_1$. In either case, we let $I_2 := [a_2, b_2]$; then we have $I_2 \subset I_1$ and $f(a_2) > 0, f(b_2) < 0$.

We continue the bisection process. Suppose that the intervals $I_1, I_2, \ldots, I_k$ have been obtained by successive bisection in the same manner. Then we have $f(a_k) > 0$ and $f(b_k) < 0$, and we set $p_k := \frac{1}{2}(a_k + b_k)$. If $f(p_k) = 0$, we take $x_0 := p_k$ and we are done. If $f(p_k) > 0$, we set $a_{k+1} := p_k, b_{k+1} := b_k$, while if $f(p_k) < 0$, we set $a_{k+1} := a_k, b_{k+1} := p_k$. In either case, we let $I_{k+1} := [a_{k+1}, b_{k+1}]$; then $I_{k+1} \subset I_k$ and $f(a_{k+1}) > 0, f(b_{k+1}) < 0$.

If the process terminates by locating a point p_n such that $f(p_n) = 0$, then we are done. If the process does not terminate, then we obtain a nested sequence of closed bounded intervals $I_n := [a_n, b_n]$ such that for every $n \in \mathbb{N}$ we have

$$f(a_n) > 0 \quad \text{and} \quad f(b_n) < 0.$$

Furthermore, since the intervals are obtained by repeated bisection, the length of I_n is equal to $b_n - a_n = (b-a)/2^{n-1}$. It follows from the Nested Intervals Property that there exists a point x_0 that belongs to I_n for all $n \in \mathbb{N}$. Since $a_n \le x_0 \le b_n$ for all $n \in \mathbb{N}$ and $\lim(b_n - a_n) = 0$, it follows that $\lim(a_n) = x_0 = \lim(b_n)$. Since f is continuous at x_0, we have

$$\lim(f(a_n)) = f(x_0) = \lim(f(b_n)).$$

The fact that $f(a_n) > 0$ for all $n \in \mathbb{N}$ implies that $f(x_0) = \lim(f(a_n)) \ge 0$. Also, the fact that $f(b_n) < 0$ for all $n \in \mathbb{N}$ implies that $f(x_0) = \lim(f(b_n)) \le 0$. Thus, we conclude that $f(x_0) = 0$. Consequently, x_0 is a root of f.

Method 2:

Proof. Let $E := \{x \in [a,b] : f(x) > 0\}$. By the continuity of f on $[a,b]$, there exist a', b' with $a < a' < b' < b$ such that

$$f(x) > 0 \ge f(u), \quad \text{for any } x \in [a, a'], \quad \text{for any } x \in (x_0, b).$$

Then $[a, a'] \subseteq E$ and thus E is non-empty.

Let

$$x_0 = \sup E \in [a', b'],$$

which exists by the completeness of $\mathbb{R}$. We claim that $f(x_0) = 0$.

By the definition of $\sup E$, there exists an increasing sequence (x_n) in E convergent to x_0, and a decreasing sequence (u_n) in $(x_0, b] \setminus E$ convergent to x_0 (eg. $u_n = x_0 + \frac{1}{n}$ with all large enough n such that $x_0 + \frac{1}{n} < b$).

Then, we have $f(x_n) > 0 \geq f(u_n)$, and it follows that

$$f(x_0) = \lim_{n\to\infty} f(x_n) \geq 0 \geq \lim_{n\to\infty} f(u_n) = f(x_0).$$

This proves that $f(x_0) = 0$. Finally, $x_0 \neq a, b$ since f is nonzero at the endpoints, so $a < x_0 < b$.

Question 4. State (without proof) all results/theorems that you have used in your answers for Q3.

The Bolzano-Weierstrass Theorem: A bounded sequence of real numbers has a convergent subsequence.

Sequential Criterion for Continuity: A function $f : I \to \mathbb{R}$ is continuous at the point $c \in I$ if and only if for every sequence (x_n) in I that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.

Nested Intervals Theorem: If $I_n = [a_n, b_n], n \in \mathbb{N}$, is a nested sequence of closed bounded intervals, then there exists a number $\xi \in \mathbb{R}$ such that $\xi \in I_n$ for all $n \in \mathbb{N}$.

Order preserving for limits of sequence: Let (x_n) be a convergent sequence in $\mathbb{R}$. If $x_n \geq 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n \geq 0$. Also, if $\alpha < \lim_n y_n < \beta$ then there exists $N \in \mathbb{N}$ such that $\alpha < y_n < \beta$ for all $n \geq N$.